The Subspace Flatness Conjecture and Faster Integer Programming

Victor Reis (Microsoft Research)

Joint work with Thomas Rothvoss (University of Washington)

MIP 2025

June 4th

Linear Programming

- Solution $x \in \mathbb{R}^n$ to a system of linear inequalities in n variables:

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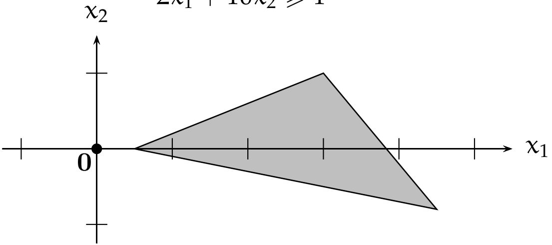

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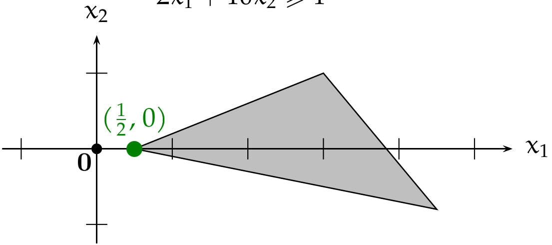

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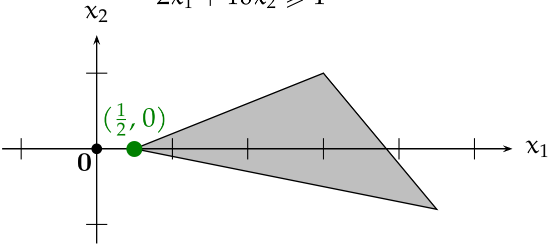

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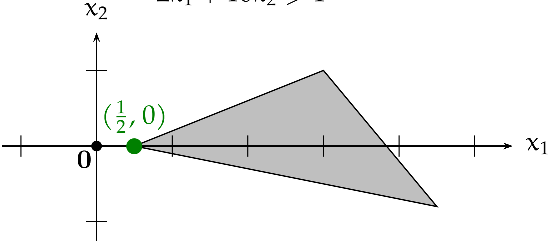

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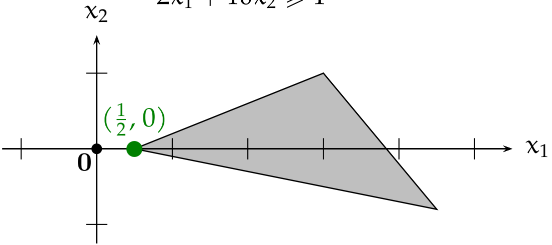

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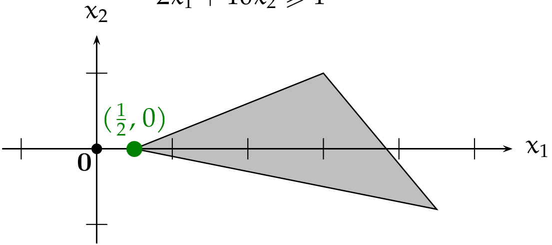

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- Polynomial time: ellipsoid method $O(n^6)$ [Khachiyan '79]

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- $O(\log n)^{4n}$ [R., Rothvoss '23]

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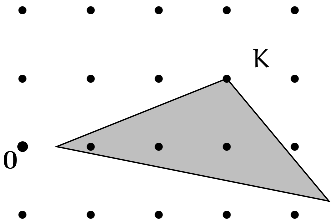

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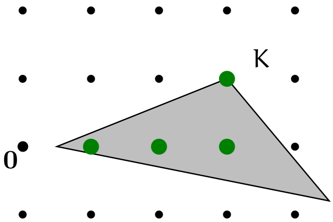

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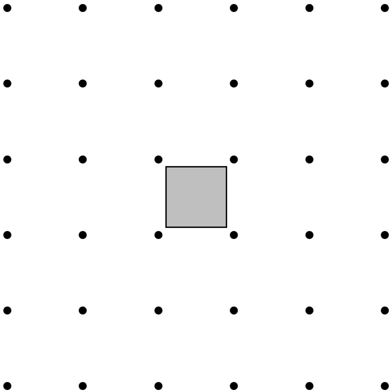

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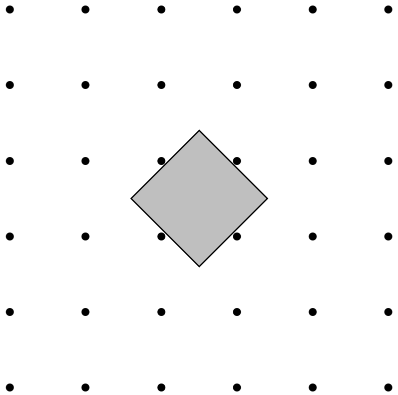

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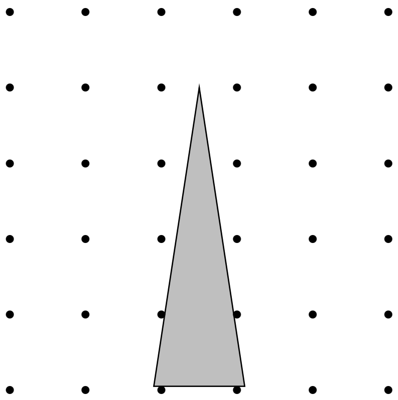

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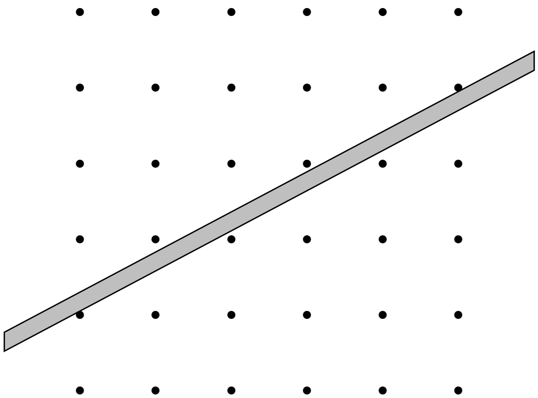

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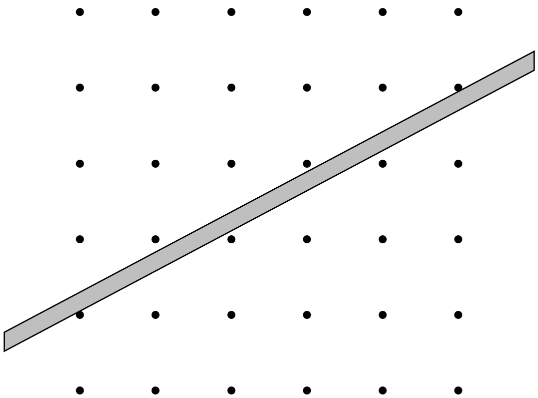

It has to be *flat*!

Lenstra's algorithm

Khintchine's flatness theorem (1947)

Given a convex $K \subset \mathbb{R}^n$, there exists either

- an integer point in K or
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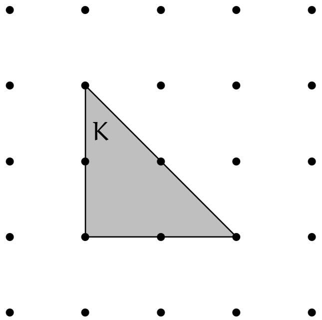

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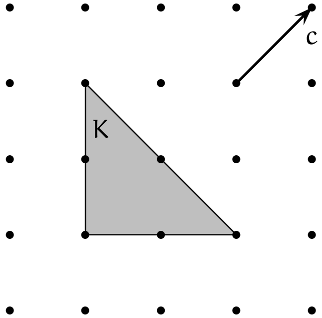

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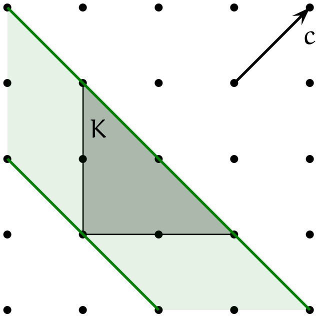

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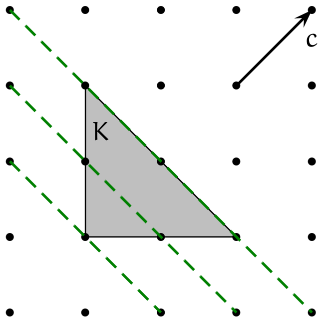

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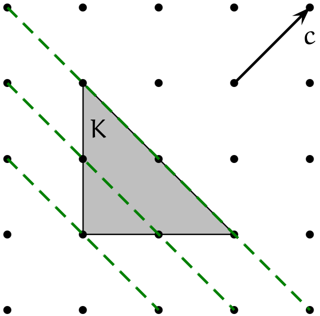

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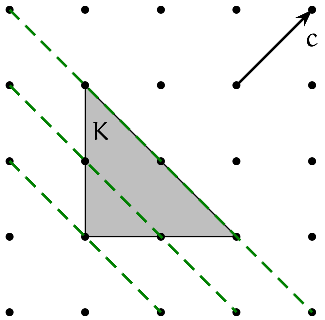

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- Lenstra's algorithm: $T(n) \leqslant T(n-1) \cdot 2^{O(n^2)} \implies T(n) \leqslant 2^{O(n^3)}$.

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Theorem [Dadush, Peikert, Vempala 2011]

We can find a direction minimizing $\max_{x \in K} c^\top x - \min_{x \in K} c^\top x$ in time $2^{O(n)}$, and solve IP in time $O(\text{Flat}(n))^n$.

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- Barrier: $\mathrm{Flat}(n) \geqslant n$
- $\mathrm{Flat}(n) \leqslant O(n \cdot (\log n)^3)$ [R., Rothvoss '23]

Daniel's vision

Can we move past hyperplane flatness with *subspaces?*

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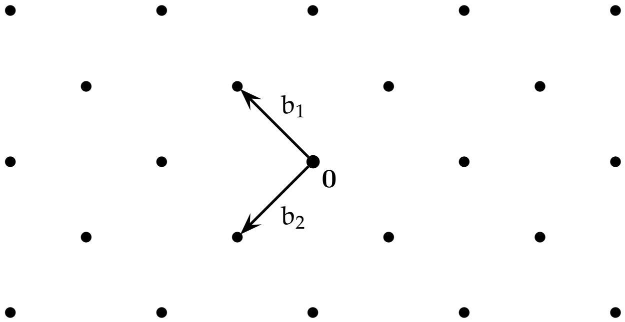
- While $\mathrm{Flat}(n) \geqslant n$, for *subspace flatness* $\geqslant \log n$ [Kannan-Lovász '88]

Lattices

- A lattice $\mathcal{L} := B\mathbb{Z}^n$ (integer linear combinations of a basis)



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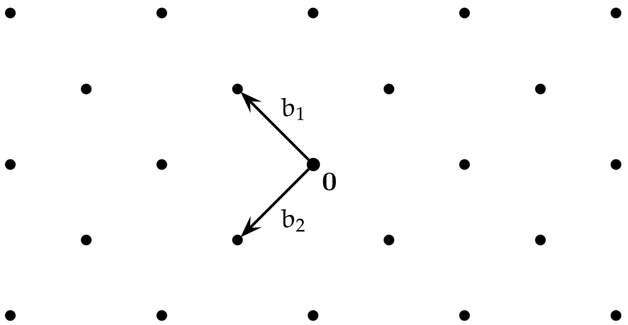

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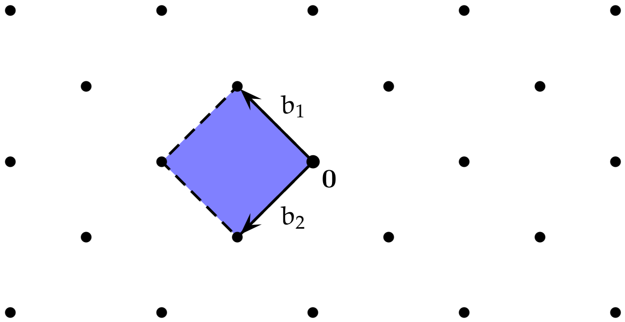

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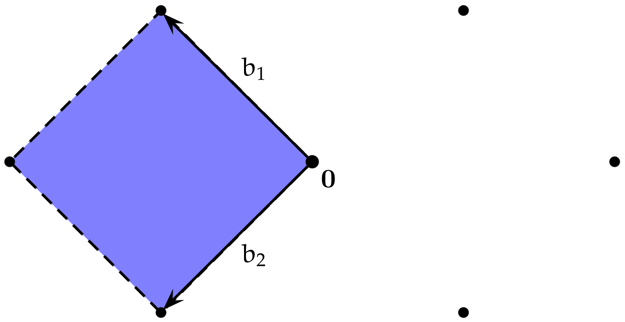

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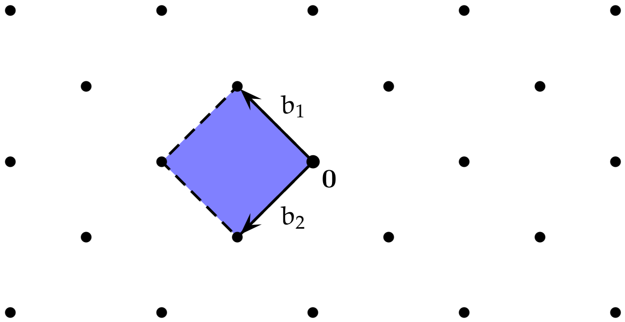

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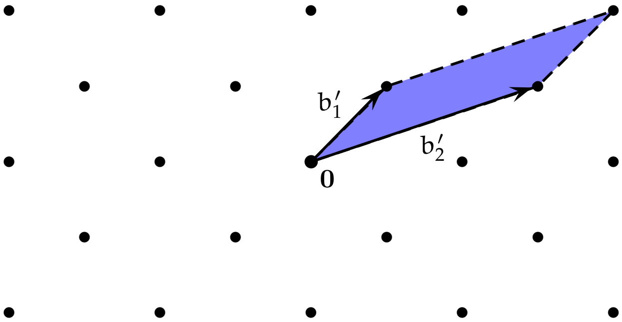

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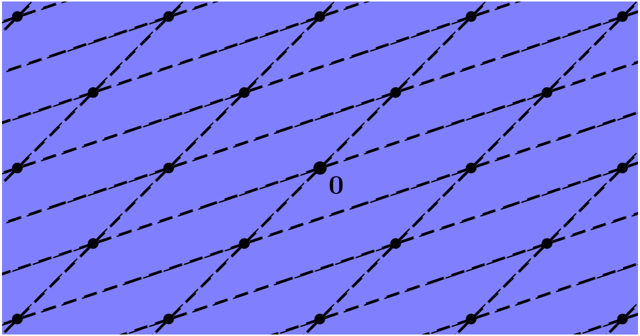

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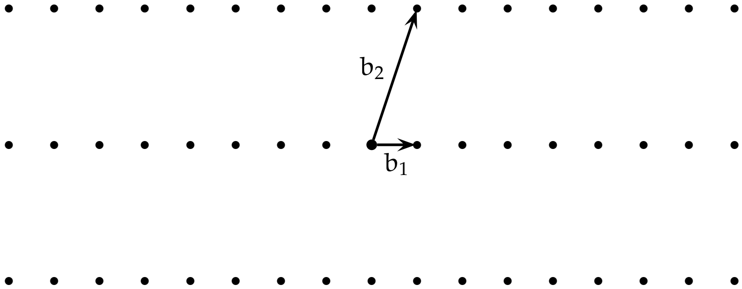
- $B[0,1]^n$ tiles $\mathbb{R}^n$ for any basis B

Covering radius

- For convex $K \subset \mathbb{R}^n$ and lattice $\mathcal{L} := B\mathbb{Z}^n$, the **covering radius** is

$$\mu(\mathcal{L}, K) := \min\{r > 0 : \mathcal{L} + r \cdot K = \mathbb{R}^n\}$$



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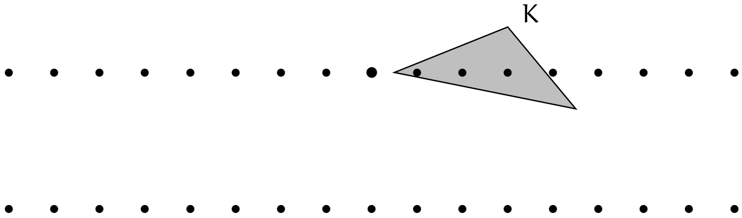

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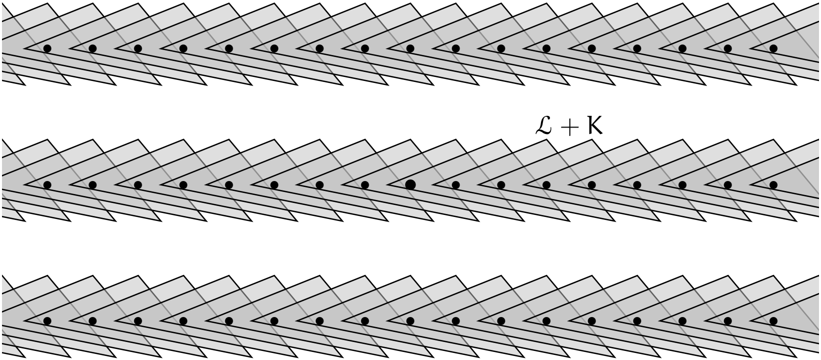

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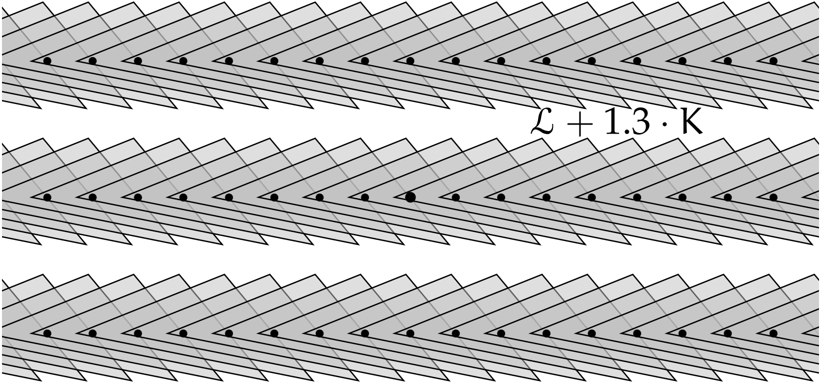

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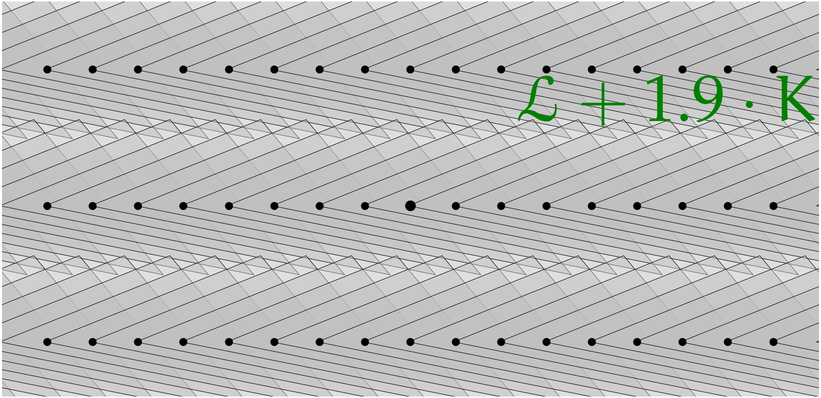

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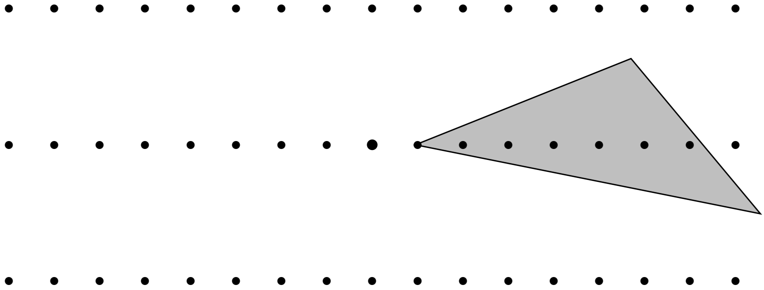

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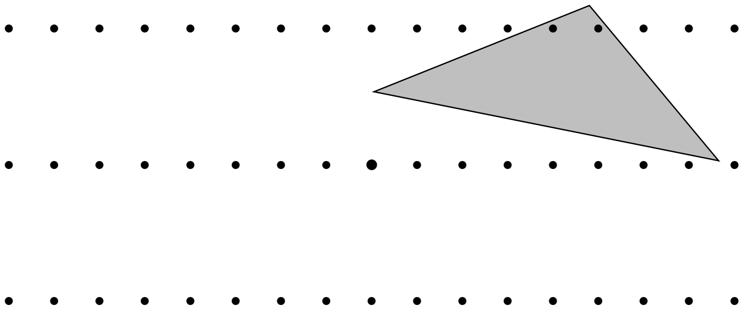

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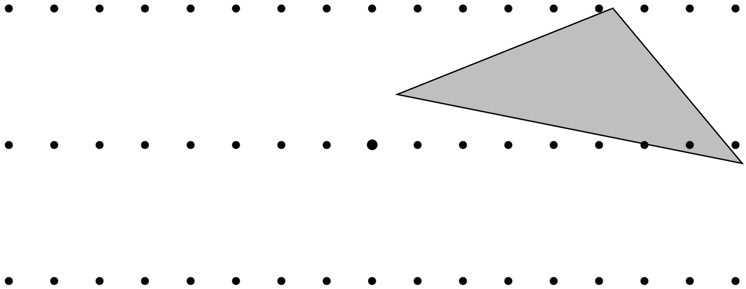

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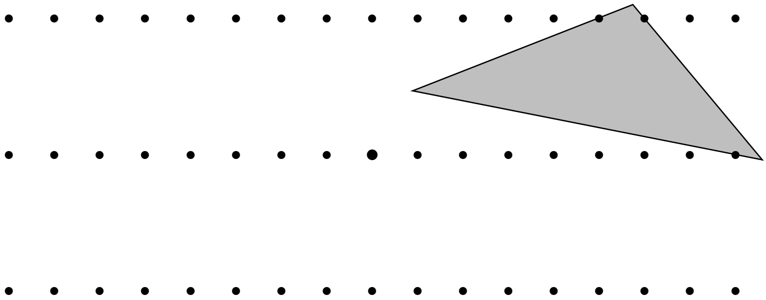

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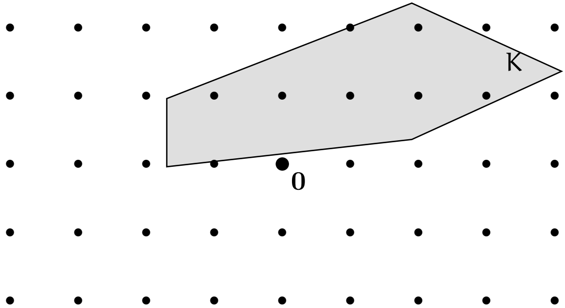

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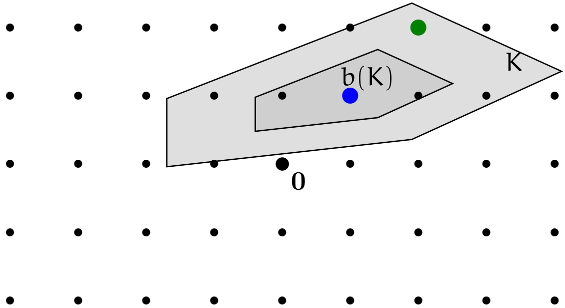

Theorem [Dadush '12]

There exists a $2^{O(n)}$ time algorithm which either:

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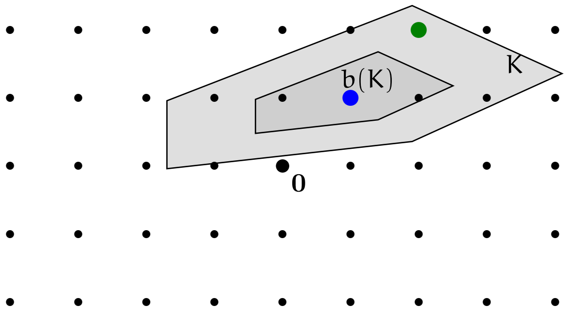

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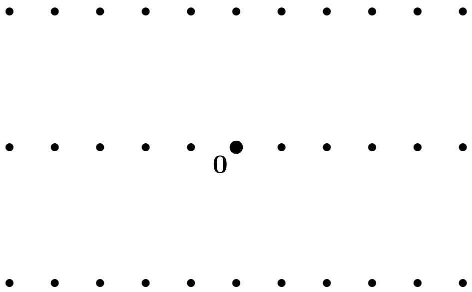

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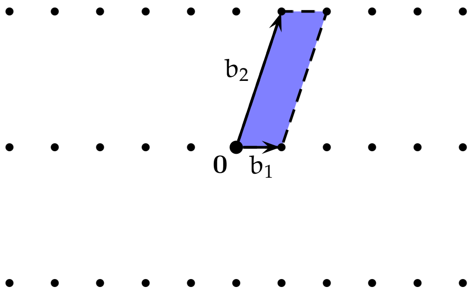

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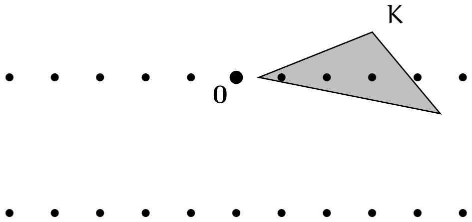

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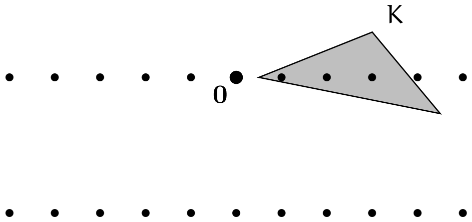

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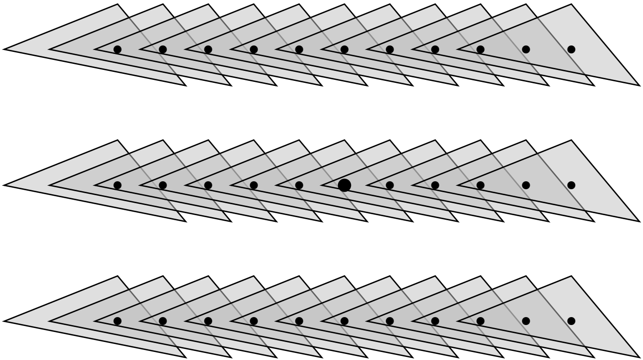

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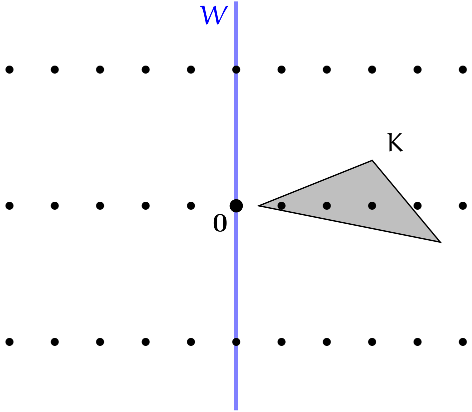

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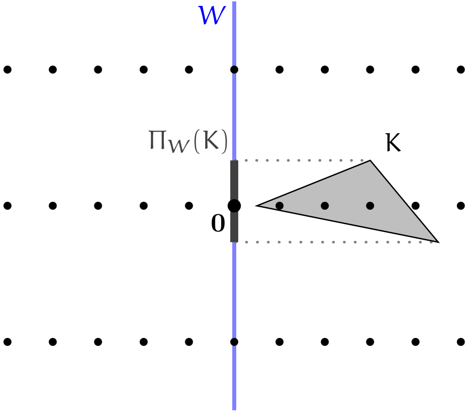

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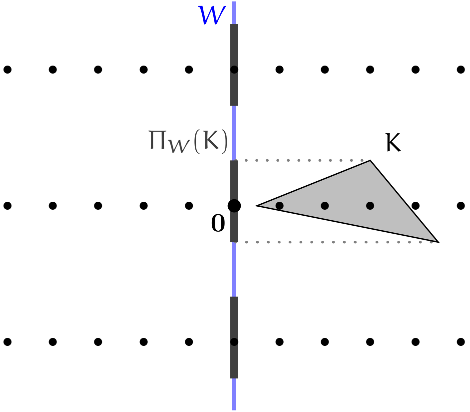

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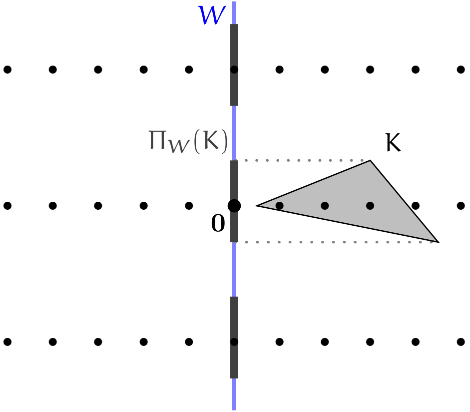

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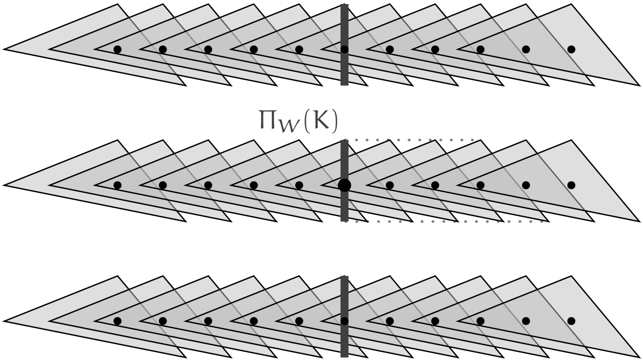

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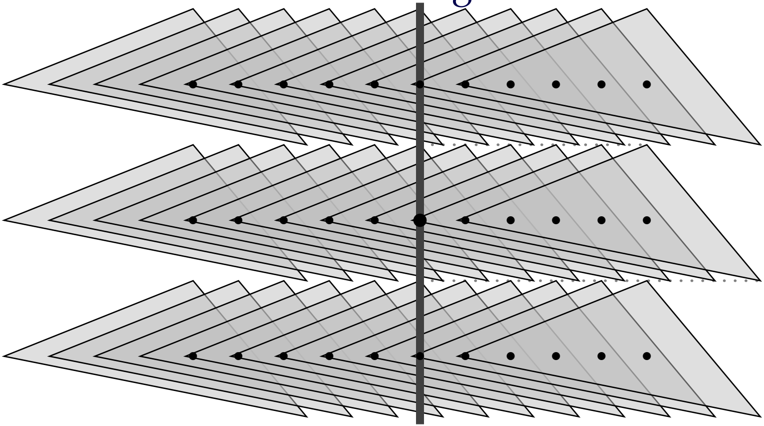

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Main results

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Corollary [R., Rothvoss '23, following Dadush '12, '19]

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For any convex $K \subset \mathbb{R}^n$ there exists an affine linear map T so that

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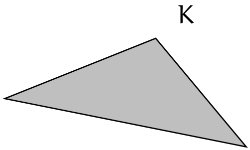

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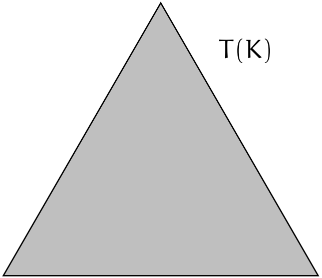

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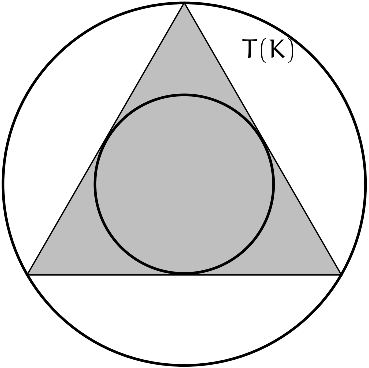

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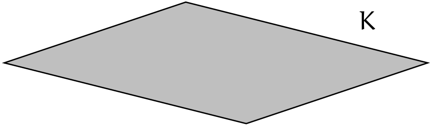
$$\Pr_{c \sim S}[c \in T(K)] > 0.9 \quad \text{and} \quad \Pr_{c \sim S}[c^\top x \leqslant O(\log n)\ \forall x \in T(K)] > 0.9$$



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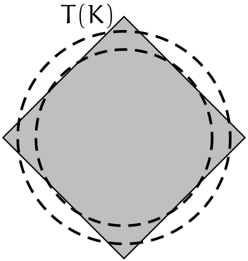

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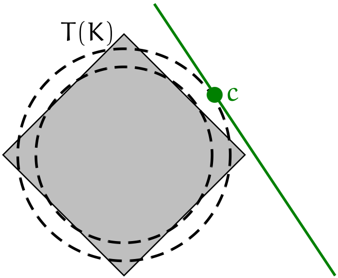

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$$\mu(\mathcal{L}, K) \leqslant \mu(\mathcal{L} \cap W, K \cap W) + \mu(\Pi_{W^\perp}(\mathcal{L}), \Pi_{W^\perp}(K))$$

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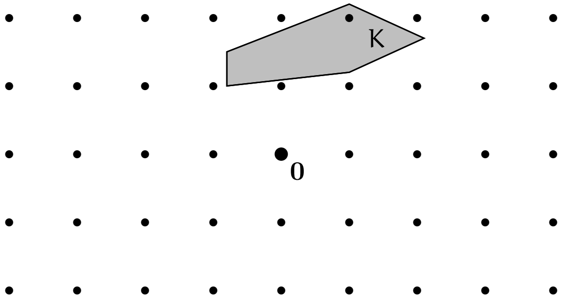
- Third $\log n$: reverse Minkowski theorem [RSD '16]

Daniel's algorithm



Input: $K \subset \mathbb{R}^n$ and lattice $\mathcal{L}$
Output: Point in $K \cap \mathcal{L}$

(1) If $\mu(\mathcal{L}, K) \leqslant \frac{1}{2}$, 2-approx. IP finds a point in $K \cap \mathcal{L}$. Else $\mu(\mathcal{L}, K) > \frac{1}{2}$
(2) Find subspace W (approximately) attaining $\alpha(\mathcal{L}, K)$
(3) Enumerate $X := \Pi_W(K) \cap \Pi_W(\mathcal{L})$
(4) Recurse on $K \cap \Pi_W^{-1}(x)$ for each $x \in X$

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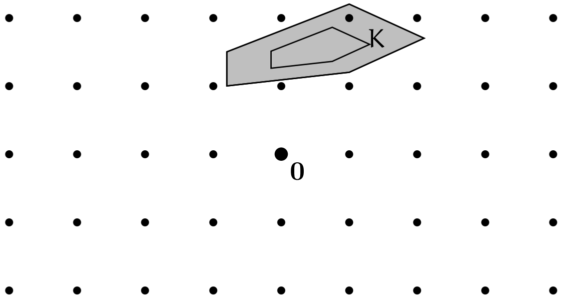

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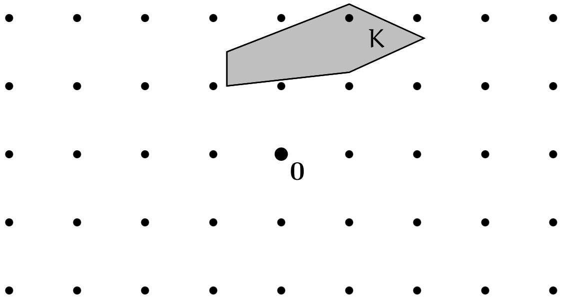

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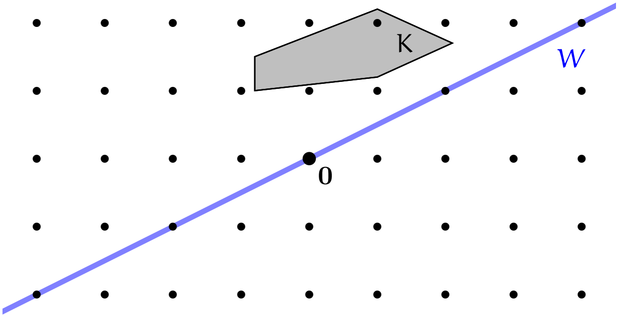

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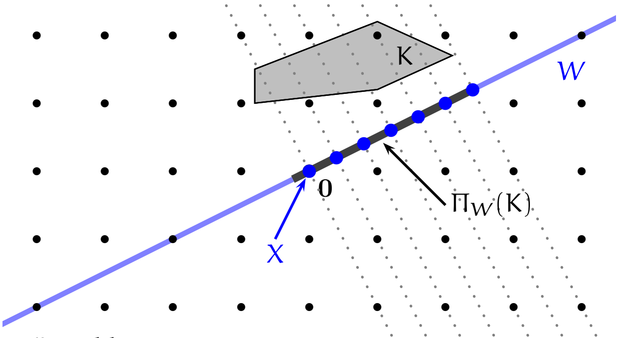

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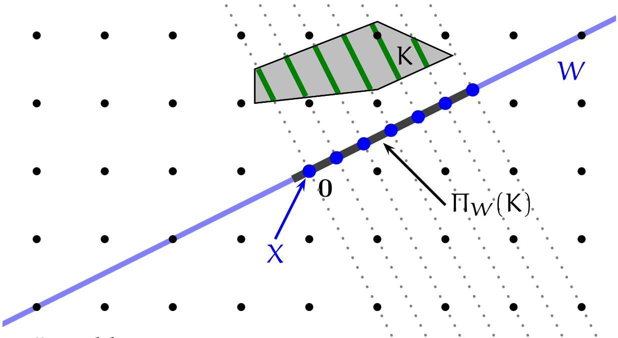

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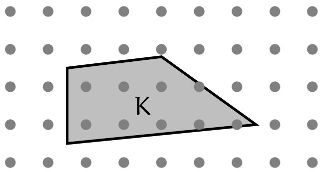
Upper bound on the number of lattice points

Theorem (Dadush 2012)

For any full rank lattice $\mathcal{L} \subseteq \mathbb{R}^n$ and convex body $K \subseteq \mathbb{R}^n$ we have

$$|K \cap \mathcal{L}| \leqslant N := 2^n \max\{\mu(\mathcal{L}, K)^n, 1\} \cdot \frac{\mathrm{vol}(K)}{\det(\mathcal{L})}.$$

Moreover, we can enumerate the points in same time.



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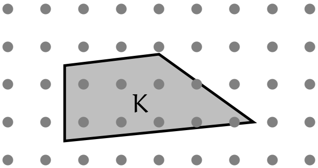

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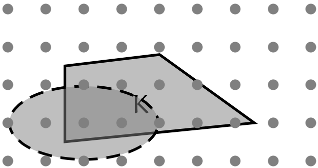

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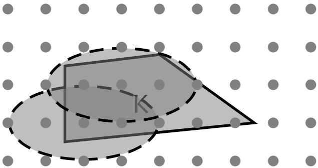

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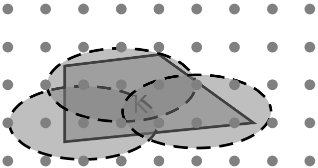

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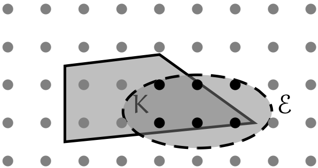

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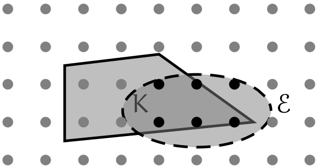
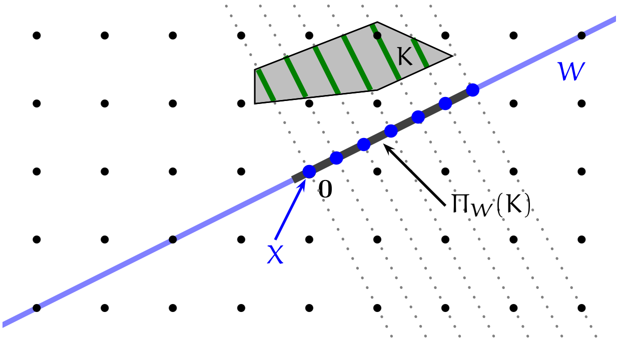
- Bound holds for any translate of K
- Any M-ellipsoid $\mathcal{E}$ also has $|\mathcal{E} \cap \mathcal{L}| \leqslant 2^{O(n)} N$
- Hence can enumerate all points in K in time $2^{O(n)} N$.



Daniel's algorithm (analysis)



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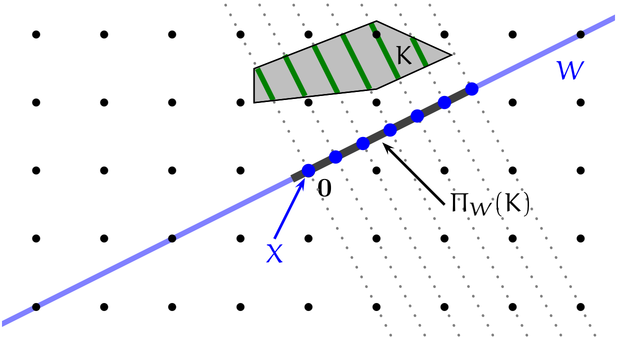

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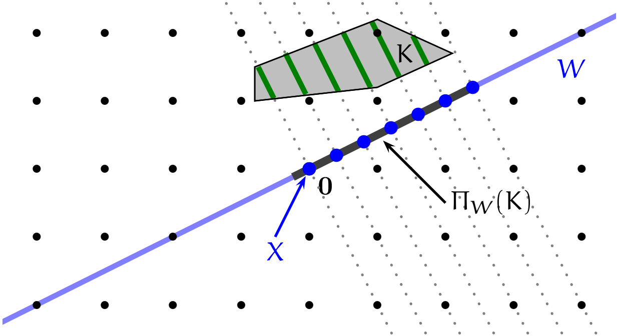

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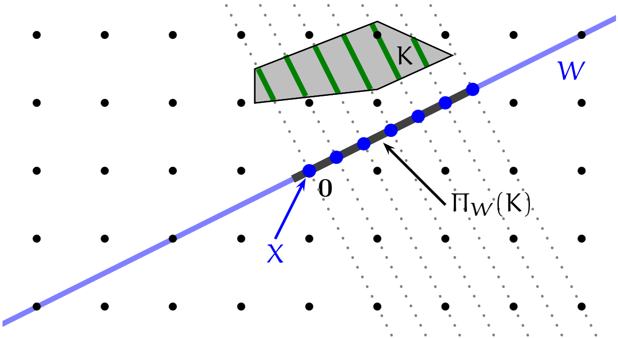

Analysis:

- Can find $(\log n)$-approximate W in time $2^{O(n)}$ [Dadush '12, '19]
- Recursion for runtime $(d := \dim W)$:

$$T(n) \leqslant 2^{O(n)} + (\log n)^{4d} \cdot T(n-d) \implies \boxed{T(n) \leqslant O(\log n)^{4n}}$$

Future directions

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Thanks for your attention!